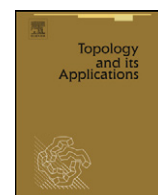




Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Ken Kunen, the set-theoretic topologist

István Juhász¹

Alfréd Rényi Institute of Mathematics, Budapest, Hungary

ARTICLE INFO

Keywords:

Set-theoretic topology
Martin's axiom
S- and L-spaces

ABSTRACT

In this short note I attempt to describe some of the most important contributions of Ken Kunen to set-theoretic topology together with some of the highlights of our personal relationship.

© 2011 Elsevier B.V. All rights reserved.

For a long time, I thought that I first met Ken in 1969 at a memorable ASL summer meeting in Manchester, England. Quite recently, however, I found out from Ken that he didn't attend that meeting. That was the meeting where I first heard about Martin's Axiom, and I firmly remember that afterwards, Dana Scott, who had been Ken's Ph.D. advisor and who was visiting Amsterdam, where I spent the year 1969, told me that the axiom could just as well be called Kunen's Axiom because Ken invented it independently of Tony Martin.

I also remember very well that soon after this I corresponded with Ken about topological applications of Martin's Axiom. His advice helped me improve the chapter on Martin's Axiom in my first tract, *Cardinal Functions* [6], written during my stay at the Mathematical Center in Amsterdam. I think this story nicely illustrates Ken's presence and influence in set-theoretic topology.

We actually (first?) met at another great meeting, the Cambridge Set Theory Summer School in 1971, and that was when we did our first joint paper [7]. In this we proved that for every cardinal κ there is a Hausdorff space of density κ and of weight $\beth_3(\kappa)$. In the proof, Ken's mastery of ultrafilters was essential.

A much closer personal contact and collaboration started when I spent the full 1974–1975 academic year at the UW Madison. Although, I think, my invitation to Madison was mainly arranged by Mary Ellen Rudin, Ken was also a gracious host. For instance, he took care of finding a place for us to stay, in the same building where he lived.

I think it is fair to say that in the mid and late 70s most activity in set-theoretic topology was concentrated around the problem of S- and L-spaces and Ken played a leading role in this activity. Recall that a regular space is an S-space if it is hereditarily separable (HS) but not hereditarily Lindelöf (HL), and is an L-space if it is HL but not HS. A strong S- (respectively, L-) space is one whose finite powers are all S- (respectively, L-) spaces. Ken proved in [10] that no strong S- or L-spaces exist under MA_{ω_1} . Everybody thought then that actually MA_{ω_1} implies no S- and L-spaces exist, but the story turned out to be much more devious.

In the summer of 1978, a topology conference was held in Budapest, where Ken presented his result on the existence of weak P-points; for more on this, see van Mill's piece in the present volume. Shortly before this meeting, Szentmiklóssy proved in [20] that both S- and L-spaces may exist under MA_{ω_1} ; he also proved, however, that no compact S-spaces may. (The same about compact L-spaces had been known before [5].) A couple of years later, Todorčević showed that PFA implies that there are no S-spaces (see [19]), and only quite recently did J. Moore prove in [17] that L-spaces exist in ZFC.

But this is only one side of the story. On the other side, consistent examples of S- and L-spaces with all sorts of nice topological properties were constructed. One of the most celebrated is the so-called Kunen line constructed from CH in [8]; this S-space is first countable and locally compact.

E-mail address: juhasz@renyi.hu.¹ The author was supported by OTKA grants Nos. 61600 and 68262.

The first construction of a compact S-space, from \diamond , is due to Fedorchuk (see [3]), and Ken came up with a greatly simplified and powerful version of this construction. He distributed this in the form of handwritten notes, as he did with many other results of his. These handwritten notes proved to be a great help for a lot of us in those pre-computer times.

Fedorchuk's compact S-space, just like Ken's simplified version of it, is the limit of an inverse system of length ω_1 . Some ideas of this inverse limit construction were then used by Ken in [11] to get a dual result, namely, a compact L-space from CH. This was quite a surprise because the existence of the “classical” compact L-space, the Suslin segment, was already known not to follow from CH.

The main idea of Ken's construction was to get the HL property of his compactum from the following ‘almost Luzin’ condition: all nowhere dense subspaces are second countable. This in turn is assured by building a measure on the space such that nowhere dense sets are null sets and null sets are second countable. The ideas of these inverse limit constructions of compact S- and L-spaces have been used later by Ken and collaborators in a number of other constructions that we cannot go into for lack of space.

Returning to the Kunen line, I recall that soon after our paper was finished, Ken sharpened the CH construction to obtain a version that is even a strong S-space. Again, this was distributed in handwritten form. A few years later he noticed that the strong Kunen line yields a compact space K such that the Banach space $C(K)$ of continuous functions on K , although non-separable, has the property that every $F \subset C(K)$ with $|F| = \aleph_1$ has an element $f \in F$ that is in the closed convex hull of $F \setminus \{f\}$. In particular, then $C(K)$ has no uncountable bi-orthogonal system. For more on this, see e.g. [18].

Another interesting application of a strong Kunen line L is given in [1]. First it is shown that a natural variation on the Alexandrov duplicate construction applied to L yields a first countable compact strong S-space X that is also scattered and hence zero-dimensional. Then they apply the theorem of Dow and Pearl from [2] to conclude that X^ω is a homogeneous strong S compactum. This is very sharp because a compact topological group is dyadic and hence cannot be an S-space. We also note here that if X is compact and X^2 is HL then X is second countable, and hence there are no compact strong L-spaces.

One more paper of Ken concerning S- and L-spaces (joint with van Douwen) that I want to mention is [16]. In this they show that a simple combinatorial statement about $\mathcal{P}(\omega)$ is equivalent to the existence of both an S-subspace and an L-subspace of the Vietoris hyperspace of a countable discrete space. Moreover, they show that CH implies this statement, and thus they provide a very short and elegant way to prove that CH implies the existence of first countable S- and L-spaces.

Leaving, but by no means exhausting, Ken's work on S- and L-spaces I must emphasize that this work, no matter how significant, is only a relatively small fraction of his activity in set-theoretic topology. I only mention box products and topological games as other areas in which Ken produced significant results. However, I do want to bring up in some detail a few more results of his that I personally like and value.

One of my favorites is his joint result with Judy Roitman on the $\sup = \max$ problem concerning spread from [15]. This says that, for any singular cardinal λ of countable cofinality, there is a Hausdorff space of spread λ but with no discrete subspace of size equal to λ if and only if there is a set of reals S of size λ such that every subset of S of size λ is non-meager. In particular, if $\lambda > \mathfrak{c}$, the cardinality of the continuum, then there is no such space.

The next is his ZFC example of a linearly Lindelöf but non-Lindelöf locally compact Hausdorff space from [12], which solved a problem of Arhangel'skii and Buzyakova. (I proudly admit to have communicated this problem to Ken.) It is easy to see that the existence of such a space is equivalent to the existence of a point x of uncountable character in a compactum X such that if a (transfinite) sequence converges to x in X then its length is cofinal with ω . Ken improved this result in [13] by proving that for every singular λ of countable cofinality there is such a point x in a compactum X with $\chi(x, X) = \lambda$. The construction uses this time a “short” inverse limit, one of length ω , and is another witness to his brilliant mastery of ultrafilters on large sets. The improved version also involves some nice tricks from model theory.

Interestingly, the global Arhangel'skii–Buzyakova question, namely, is there a non-first countable compactum in which the length of every (non-trivial) converging sequence is ω -cofinal, has a negative answer if $\mathfrak{c} < \aleph_\omega$ (see [4]). One of my favorite problems at present asks if a negative answer is provable in ZFC, but this is still waiting for Ken.

The last paper I have to mention is [14], which vastly simplifies the proof of, and at the same time improves, the main result of [9]. The latter says that, after adding any number of Cohen reals to our ground model V , in the extension every compact scattered space has at most \mathfrak{c}^V many Cantor–Bendixson levels of size ω . Now, Ken shows that the same conclusion holds in any ccc generic extension obtained either by adding any number of random reals or by forcing with a finite support power of a (necessarily ccc) partial order of cardinality at most \mathfrak{c} . The beauty of Ken's paper lies not so much in that it solves a problem raised in our paper, namely whether our Cohen reals result can also be obtained by adding random reals, but in that his powerful proof is much simpler than ours. I recommend you read it.

I cannot resist, closing this tribute to Ken, to mention here another problem that, I dare say, is one of the most intriguing problems in set-theoretic topology today: Is it consistent to have a compact scattered space that has \aleph_3 Cantor–Bendixson levels of size ω ?

References

- [1] R. de la Vega, K. Kunen, A compact homogeneous S-space, *Topology Appl.* 136 (2004) 123–127.
- [2] A. Dow, E. Pearl, Homogeneity in powers of zero-dimensional first countable spaces, *Proc. Amer. Math. Soc.* 125 (1997) 2503–2510.
- [3] V.V. Fedorchuk, Fully closed mappings and the consistency of ..., *Mat. Sb.* 99 (1976) 3–33.
- [4] I. Gorelic, I. Juhász, AB-compacta, *Comment. Math. Univ. Carolin.* 49 (1) (2008) 141–146.

- [5] I. Juhász, Martin's axiom solves Ponomarev's problem, *Bull. Acad. Pol. Sci.* 18 (1970) 71–74.
- [6] I. Juhász, *Cardinal Functions in Topology*, Math. Centre Tracts., vol. 34, Math. Centrum, Amsterdam, 1971.
- [7] I. Juhász, K. Kunen, On the weight of Hausdorff spaces, *General Topology Appl.* 3 (1973) 47–49.
- [8] I. Juhász, K. Kunen, M.E. Rudin, Two more hereditarily separable non-Lindelöf spaces, *Canad. J. Math.* 28 (1976) 998–1005.
- [9] I. Juhász, S. Shelah, L. Soukup, Z. Szentmiklóssy, Cardinal sequences and Cohen real extensions, *Fund. Math.* 181 (2004) 75–88.
- [10] K. Kunen, Strong S and L spaces under MA, in: *Set Theoretic Topology*, Proc. of Athens Spring Top. Meeting, 1976, pp. 265–268.
- [11] K. Kunen, A compact L-space under CH, *Topology Appl.* 12 (1981) 283–287.
- [12] K. Kunen, Locally compact linearly Lindelöf spaces, *Comment. Math. Univ. Carolin.* 43 (2002) 155–158.
- [13] K. Kunen, Small locally compact linearly Lindelöf spaces, *Topology Proc.* 29 (2005) 193–198.
- [14] K. Kunen, Compact scattered spaces in forcing extensions, *Fund. Math.* 185 (2005) 261–266.
- [15] K. Kunen, J. Roitman, Attaining the spread at singular cardinals of cofinality ω , *Pacific J. Math.* 70 (1977) 199–205.
- [16] K. Kunen, E. van Douwen, L-spaces and S-spaces in $\mathcal{P}(\omega)$, *Topology Appl.* 14 (1982) 143–149.
- [17] J.T. Moore, A solution to the L space problem, *J. Amer. Math. Soc.* 19 (3) (2006) 717–736.
- [18] S. Negrepointis, Banach spaces and topology, in: K. Kunen, J.E. Vaughan (Eds.), *Handbook of Set-Theoretic Topology*, North-Holland, Amsterdam, 1984, pp. 1045–1142.
- [19] J. Roitman, Basic S and L, in: *Handbook of Set-Theoretic Topology*, North-Holland, Amsterdam, 1984, pp. 295–326.
- [20] Z. Szentmiklóssy, S-spaces and L-spaces under Martin's axiom, II, in: *Coll. Math. Soc. J. Bolyai*, vol. 23, North-Holland, Amsterdam, 1980, pp. 1139–1145.